

r -extension of Dunkl operator in one variable and Bessel functions of vector index

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Abstract

In this work we present an operator D_μ constructed with the help of the cyclic group set of the r^{th} roots of unity. This operator constitute an r -extension of the Dunkl operator in one variable because when $r = 2$ it reduces to the classical one and admits as eigenfunctions the Bessel functions of vector index early deeply studied by Klyuchantsev. This paper is argued by specific examples and contains the beginning of some elements of harmonic analysis related to this operator.

Keywords : Dunkl operator, Bessel functions, Fourier transform, transmutation operator.

2000 AMS Mathematics Subject Classification—Primary 33D15, 47A05.

1 Introduction

At the beginning of the last decade of the twentieth century C. Dunkl [3, 4] in a series of articles using reflection groups introduced a differential-difference operator now commonly called Dunkl operator and became a great center of interest and inspiration in many areas of pure and applied mathematics. This operator has generated a rich harmonic analysis developed by several authors. and involves a combination of Bessel functions of index α as eigenfunctions . So exploiting specific properties of these well-known special functions great analysis and an armada of applications was born.

Having knowledge of the progress of this topic in many scientific areas, we are always asked and highly intrigued by his extension in higher r -order which involves Bessel functions of index vector j_μ which are eigenfunctions of Δ_μ , $\mu = (\alpha_1, \dots, \alpha_{r-1})$ a differential operator of order r . These last functions are one of the generalized Bessel functions mentioned by Watson in his venerable book [12] and greatly studied with applications by many authors (see [1, 6, 7, 8, 9]) and the references therein . The cyclic group $C_r = \{\omega^r = 1, \quad r = 1, 2, 3, \dots\}$ plays a central role in the definition of our r -extension particularly for define the r -even and the r -odd functions of order $l = 1, \dots, r - 1$ and leads to decompose the space of functions in direct sum of invariant subspaces with appropriate projectors T_k , $k = 1, \dots, r - 1$ useful to construct D_μ the r -extension of the Dunkl operator having as fundamental property $D_\mu^r = \Delta_\mu$ on a particular subspace F_k . In addition to the construction process of the operator D_μ , we give a particular interest in many cases especially in all the paragraphs discussed namely representation integral, associated Riemann-Liouville transform, transmutation and r -extension Dunkl transform.

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2 The operator D_μ

Throughout this paper r is an integer great than 1, $\omega = e^{\frac{2i\pi}{r}}$ and we put

$$C_r = \{1, \omega, \omega^2, \dots, \omega^{r-1}\}$$

the cyclic group of order r . We denote by F the space of complex valued functions and we consider on this space the following actions

$$s_k g(x) = \omega^k g(\omega x), \quad k = 0, 1, 2, \dots$$

Putting F_k the subspace of F invariant by s_k ; namely

$$g \in F_k \Leftrightarrow s_k g = g. \quad (1)$$

Now we introduce the collection of the projector operators defined by the relations

$$T_k = \frac{1}{r} \sum_{n=0}^{r-1} s_k^n; \quad k = 0, 1, 2, \dots \quad (2)$$

which is slightly different from that introduced by [10].

Very often in mathematical literature [8, 9], the function $T_0 g$ is called the r -even part of g and the functions $T_k g$, $r = 1, 2, \dots, r-1$ are called r -odd of order k of g .

Taking account of the fact that $s_k^r = id$, one can easily show that the following properties hold:

1. The operators T_k and s_k , $k = 0, 1, 2, \dots$ commute in the the sense

$$T_k s_k = s_k T_k$$

2. The subspace F_k can be also characterized as:

$$g \in F_k \Leftrightarrow T_k g = g.$$

3. We have

$$k \neq l \Leftrightarrow T_k T_l = 0.$$

Hence we have $T_k^2 = T_k$ so that the operators T_k are projectors and a simple computation leads to that

$$F = F_1 \oplus \dots \oplus F_{r-1}.$$

For clarity and since their importance and their interference in the demonstrations we summarize here the following properties which are useful in the sequel.

1. The derivative operator $\frac{d}{dx}$ maps the space F_k into F_{k+1} .

2. The multiplication operator by $\frac{1}{x}$ satisfies

$$\frac{1}{x}s_k = s_{k+1}\frac{1}{x}$$

and it maps the space F_k into F_{k+1} . Moreover

$$\frac{1}{x}T_k = T_{k+1}\frac{1}{x}$$

3. For a real, we denote by L_a the class of operators given by

$$L_a(f) = x^{-a}\frac{d}{dx}(x^a f) = f' + \frac{a}{x}f. \quad (3)$$

Firstly we have $x^{-b}L_a x^b = L_{a+b}$ and L_a maps F_k into F_{k+1} .

Taking into account the concepts and properties presented earlier now it seems appropriate to introduce the following definitions.

Definition 1 Let $\mu = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$ be a vector of \mathbb{R}^r . We define the Bessel operator of order r associated to the index vector μ by

$$\Delta_\mu = L_{a_{r-1}} \circ \dots \circ L_{a_0}$$

where we have put

$$a_k = r\alpha_k + k \quad k = 0, 1, \dots, r-1, \quad (4)$$

and L_a is the operator given by (3).

Definition 2 We define the r -extension of Dunkl operator by

$$D_\mu = \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k$$

where the coefficients a_k and the operators T_k are given respectively by (4) and (2).

Before any thing let us justify the appellation by the following proposition

Proposition 1 For $f \in F_k$, $k = 0 \dots (r-1)$, we have

$$D_\mu^r(f) = \Delta_\mu(f).$$

Proof. This result is first a consequence of the fact that D_μ maps the F_k in F_{k+1} because the operators $\frac{d}{dx}$ and T_k have the same properties. Since if $g \in F_k$ then $D_\mu g = L_{a_k} g$ we deduce then $D_\mu g = L_{a_k} g \in F_{k+1}$, hence

$$D_\mu^2 g = L_{a_{k+1}} L_{a_k} g \in F_{k+2}.$$

By induction and the fact that $F_{k+r} = F_k$ we find

$$\text{if } g \in F_k \Rightarrow D_\mu g = \Delta_\mu g.$$

■

Having defined the operators Δ_μ and D_μ , it is quite natural to seek their eigenfunctions. For this we introduce the Bessel functions of vector index

$$\mu = (\alpha_0, \alpha_1, \dots, \alpha_{r-1}) \in \mathbb{R}^r \setminus \mathbb{Z}_-^r$$

by

$$j_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(\alpha_0 + 1)_n (\alpha_1 + 1)_n \dots (\alpha_{r-1} + 1)_n} \frac{x^{nr}}{r^{nr}},$$

where $(\beta)_n = \frac{\Gamma(\beta+1)}{\Gamma(\beta)}$.

The knowledgeable reader should note that this function differs from that studied in [9] by the number of components of the index vector.

The above series is entire and taking account of

$$\Delta_\mu x^{rn} = r^n (\alpha_0 + n)(\alpha_1 + n) \dots (\alpha_{r-1} + n) x^{(n-1)r},$$

one can state :

Proposition 2 *For a complex λ we have*

$$\Delta_\mu j_\mu(\lambda x) = -\lambda^r j_\mu(\lambda x);$$

with $j_\mu(0) = 1$.

Now we put

$$\theta = e^{i\frac{\pi}{r}}$$

and we consider the r -extension function which call it also r -Dunkl kernel

$$E_\mu(x) = j_\mu(x) + \frac{1}{\theta} D_\mu j_\mu(x) + \dots + \frac{1}{\theta^{r-2}} D_\mu^{r-2} j_\mu(x) + \frac{1}{\theta^{r-1}} D_\mu^{r-1} j_\mu(x). \quad (5)$$

Proposition 3 *For complex λ the function $x \mapsto E_\mu(\lambda x)$ is an eigenfunction for the generalized Dunkl operator D_μ with $\theta\lambda$ as eigenvalue:*

$$D_\mu E_\mu(\lambda x) = \theta\lambda E_\mu(\lambda x).$$

Proof. Indeed, to be convinced it suffices to make the following computations:

$$\begin{aligned} D_\mu E_\mu &= D_\mu j_\mu + \frac{1}{\theta} D_\mu^2 j_\mu + \dots + \frac{1}{\theta^{r-2}} D_\mu^{r-1} j_\mu + \frac{1}{\theta^{r-1}} D_\mu^r j_\mu \\ &= D_\mu j_\mu + \frac{1}{\theta} D_\mu^2 j_\mu + \dots + \frac{1}{\theta^{r-2}} D_\mu^{r-1} j_\mu + \frac{\theta^r}{\theta^{r-1}} j_\mu \\ &= \theta \left[j_\mu + \frac{1}{\theta} D_\mu j_\mu + \dots + \frac{1}{\theta^{r-1}} D_\mu^{r-1} j_\mu \right] \\ &= \theta E_\mu. \end{aligned}$$

so the result follows. ■

For the rest of this work we need to compute the action of the r -extension of Dunkl operator D_μ on the Bessel functions of vector index j_μ .

Indeed using the fact that $D_\mu x^{nr} = L_{a_0} x^{nr} = r(\alpha_0 + n)x^{nr-1}$, we obtain

$$D_\mu j_\mu(x) = r \sum_{n=0}^{\infty} (-1)^n \frac{1}{(\alpha_0 + 1)_n \dots (\alpha_{r-1} + 1)_n} (\alpha_0 + n) \frac{x^{nr-1}}{r^{nr}}.$$

Then we distinguish two cases :

Case 1 : when $\alpha_0 \neq 0$.

We have

$$D_\mu j_\mu(x) = \frac{r}{x} \alpha_0 j_{\mu-1},$$

remark that we have adopt the convention : for $\mu = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$ we put

$$\mu - 1 = (\alpha_0 - 1, \alpha_1, \dots, \alpha_{r-1}).$$

Case 2 : when $\alpha_0 = 0$.

With a slice change of computation we have

$$D_\mu j_\mu(x) = -\frac{1}{(\alpha_1 + 1) \dots (\alpha_{r-1} + 1)} \left(\frac{x}{r}\right)^{r-1} j_{\mu+1}.$$

We have also adopt the convention : for $\mu = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$ we put

$$\mu + 1 = (\alpha_0, \alpha_1 + 1, \dots, \alpha_{r-1} + 1).$$

As mentioned in abstract we present in the following three explicit examples which illustrate the operators Δ_μ and D_μ .

Example 1 : $r = 2, \omega = -1, \theta = i, \mu = (0, \alpha)$.

Then

$$a_0 = 0, a_1 = 2\alpha + 1.$$

As a consequence, we have

$$\Delta_\mu = L_{2\alpha+1} L_0 = \frac{d^2}{dx^2} + \frac{2\alpha + 1}{x} \frac{d}{dx},$$

which is exactly the well known Bessel operator admitting as eigenfunction the normalized Bessel function

$$j_\alpha(x) = j_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! (\alpha + 1)_n} \frac{x^{2n}}{2^{2n}}.$$

Now since

$$T_1 g(x) = \frac{g(x) + s_1 g(x)}{2} = \frac{g(x) - g(-x)}{2}$$

we discover that the operator D_μ is the classical Dunkl operator in one variable:

$$D_\mu = D_\alpha = \frac{d}{dx} + \frac{\alpha + 1}{x} T_1,$$

and using the conventional notation introduced before namely

$$\mu + 1 = (0, \alpha + 1)$$

and the fact that $\theta = i$, we find that the eigenfunctions E_μ coincide with the classical one.

Example 2 : $r = 3, \omega = e^{\frac{i2\pi}{3}}, \theta = e^{\frac{i\pi}{3}}, \mu = (0, \alpha - \frac{1}{3}, -\frac{2}{3})$.

Taking account of the relation (4) between a_k and α_k we deduce that

$$a_0 = 0, \quad a_1 = 3\alpha, \quad a_2 = 0.$$

So

$$D_\mu = L_0 L_{3\alpha} L_0 = \frac{d^3}{dx^3} - \frac{3\alpha}{x} \frac{d^2}{dx^2} + \frac{3\alpha}{x^2} \frac{d}{dx}.$$

The previous operator was greatly studied in [8]. Its eigenfunction is given by

$$j_\mu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \left(\alpha + \frac{2}{3}\right)_n \left(\frac{1}{3}\right)_n} \frac{x^{3n}}{3^{3n}}.$$

The correspondent Dunkl operator is:

$$D_\mu = \frac{d}{dx} + \frac{3\alpha}{x} T_1$$

with

$$T_1 g(x) = \frac{g(x) + \omega g(\omega x) + \omega^2 g(\omega^2 x)}{3}.$$

We have the relation

$$D_\mu j_\mu(x) = \frac{d}{dx} j_\mu(x) = -\frac{1}{\left(\alpha + \frac{2}{3}\right)\left(\frac{1}{3}\right)} \left(\frac{x}{3}\right)^2 j_{\mu+1}(x),$$

with $\mu + 1 = (0, \alpha + \frac{2}{3}, \frac{1}{3})$. So

$$j_{\mu+1}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \left(\alpha + \frac{5}{3}\right)_n \left(\frac{4}{3}\right)_n} \frac{x^{3n}}{3^{3n}}.$$

Since $D_\mu j_\mu \in F_1$, we obtain

$$D_\mu^2 j_\mu(x) = \frac{d}{dx} D_\mu j_\mu(x) + \frac{3\alpha}{x} T_1 D_\mu j_\mu(x).$$

Direct computations give

$$D_\mu^2 j_\mu(x) = \frac{x^4}{4(3\alpha + 2)(3\alpha + 5)} j_{\mu+2}(x) - x j_{\mu+1};$$

with $\mu + 2 = (0, \mu + \frac{5}{3}, \frac{4}{3})$ and

$$j_{\mu+2}(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n! \left(\alpha + \frac{8}{3}\right)_n \left(\frac{7}{3}\right)_n} \frac{x^{3n}}{3^{3n}}.$$

Finally taking account of the above results we state that the eigenfunction of the correspondent Dunkl operator is then

$$E_{\mu}(x) = j_{\mu}(x) + e^{-\frac{i\pi}{3}} D_{\mu} j_{\mu}(x) + e^{-\frac{2i\pi}{3}} D_{\mu}^2 j_{\mu}(x)$$

$$E_{\mu}(x) = j_{\mu}(x) - \left[\frac{e^{-\frac{i\pi}{3}}}{3^2 \left(\alpha + \frac{2}{3}\right) \left(\frac{1}{3}\right)} \right] j_{\mu+1}(x) + \frac{e^{-\frac{2i\pi}{3}}}{4(3\alpha + 2)(3\alpha + 5)} x^4 j_{\mu+2}(x).$$

Remark 1 : It is easy to see that the following commutation holds:

$$L_a T_k = T_{k+1} L_a$$

and since $T_{k+r} = T_k$ this leads that the operators Δ_{μ} and T_k commute:

$$\Delta_{\mu} T_k = T_k \Delta_{\mu}.$$

Note that if g is a function such that $\Delta_{\mu} g = -g$ and since we have the unique decomposition

$$g = T_0 g + \dots + T_{r-1} g.$$

One can interpret the component $T_k g$ as the unique solution of the previous equation restraint to the subspace F_k .

Example 3 : $\mu = (0, -\frac{1}{r}, \dots, -\frac{r-1}{r})$, $\theta = e^{\frac{i\pi}{r}}$.

In this situation as $\alpha_k = -\frac{k}{r}$ we have $a_k = 0$, hence $\Delta_{\mu} = (\frac{d}{dx})^r$ and $D_{\mu} = \frac{d}{dx}$. It is clear that the function $e_{\theta}(x) = e^{\theta x}$ satisfies the equation

$$\Delta_{\mu} e_{\theta} = -e_{\theta}$$

The components $T_k e^{\theta x}$, $k = 0, 1, \dots, r-1$ are called the r -trigonometric functions [8]. We have in particular

$$\cos_r(x) = T_0 e^{\theta x} = \frac{1}{r} \sum_{k=0}^{r-1} e^{\theta \omega^k x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1)_n \left(-\frac{1}{r} + 1\right)_n \dots \left(-\frac{r-1}{r} + 1\right)_n} \frac{x^{nr}}{r^{nr}}$$

This last function is the unique eigenfunction of $\Delta_{\mu} = (\frac{d}{dx})^r$ with take the value 1 at $x = 0$. We notice that the Dunkl kernel can be written

$$\begin{aligned} E_{\mu}(x) &= \cos_r(x) + \frac{1}{\theta} D_{\mu} \cos_r(x) + \dots + \frac{1}{\theta^{r-1}} D_{\mu}^{r-1} \cos_r(x) \\ &= (T_0 + \dots + T_{r-1}) e^{\theta x} = e^{\theta x}. \end{aligned}$$

3 Integral representations

In this section we attempt to show that the functions j_μ and E_μ have some useful integral representations. For the first function the reader can find its integral representation already shown by Klyuchantsev [9] and which is recalled in the proof of Theorem 1.

The following lemma is basic and it is a consequence of properties of Euler functions.

Lemma 1 *We have*

$$r \int_0^1 (1 - u^r)^{y-1} u^{rx-1} du = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}$$

provided the integral converges.

We deduce then the following identities

$$r \int_0^1 u^{nr} (1 - u^r)^{\alpha_i + \frac{i}{r} - 1} u^{r-(i+1)} du = \frac{\Gamma(-\frac{i}{r} + 1 + n) \Gamma(\alpha_i + \frac{i}{r})}{\Gamma(\alpha_i + 1 + n)}.$$

$$r \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + \frac{i}{r}) \Gamma(-\frac{i}{r} + 1)} \int_0^1 \frac{\Gamma(-\frac{i}{r} + 1)}{\Gamma(-\frac{i}{r} + 1 + n)} u^{nr} (1 - u^r)^{\alpha_i + \frac{i}{r} - 1} u^{r-(i+1)} du = \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + 1 + n)}$$

So we have

$$r \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + \frac{i}{r}) \Gamma(-\frac{i}{r} + 1)} \int_0^1 \frac{1}{(-\frac{i}{r} + 1)_n} u^{nr} (1 - u^r)^{\alpha_i + \frac{i}{r} - 1} u^{r-(i+1)} du = \frac{1}{(\alpha_i + 1)_n}$$

and in general case

$$\begin{aligned} & \int_0^1 \dots \int_0^1 (-1)^n \frac{1}{(1)_n \dots (-\frac{r-1}{r} + 1)_n} \frac{(xu_0 \dots u_{r-1})^{nr}}{r^{nr}} w_0(u_0) \dots w_{r-1}(u_{r-1}) du_0 \dots du_{r-1} \\ &= (-1)^n \frac{1}{(\alpha_0 + 1)_n \dots (\alpha_{r-1} + 1)_n} \frac{x^{nr}}{r^{nr}}. \end{aligned}$$

Let

$$w_\mu(u) = \prod_{i=0}^{r-1} (1 - u_i^r)^{\alpha_i + \frac{i}{r} - 1} u_i^{r-(i+1)} \quad (6)$$

$$c_\mu = \prod_{i=0}^{r-1} r \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i + \frac{i}{r}) \Gamma(-\frac{i}{r} + 1)} \quad (7)$$

and

$$u_r = u_0 \dots u_{r-1}$$

$$du = du_0 \dots du_{r-1}$$

and using the function \cos_r presented in Example 3, we have

Theorem 1 *The Bessel function of vector index possess the following integral representation*

$$j_\mu(x) = c_\mu \int_{[0,1]^r} \cos_r(xu_r) w_\mu(u) du$$

where c_μ and w_μ are given respectively by (7) and (6).

Note that in this representation we can remove the components u_i associated with indices i such that $a_i = 0$. In this case the Mehler representation takes the following form

$$j_\mu(x) = c_{\mu'} \int_{[0,1]^{r'}} \cos_r(xu_{r'}) w_{\mu'}(u) du$$

with r' is the number of index i such that $a_i \neq 0$ and μ' contains only the associate α_i .

Theorem 2 *the r -Dunkl kernel possess the following integral representation*

$$E_\mu(x) = c_\mu \int_{[0,1]^r} \left(T_0 + \sum_{k=1}^{r-1} \frac{1}{\theta^k} T_k L_{a_{k-1}} \dots L_{a_0} \right) e_\theta(xu_r) w_\mu(u) du.$$

Proof. This is a consequence of definition of $E_\mu(x)$ and the identity

$$\begin{aligned} \frac{1}{\theta^k} D_\mu^k j_\mu(x) &= c_\mu \int_{[0,1]^r} \frac{1}{\theta^k} D_\mu^k T_0 e_\theta(xu_r) w_\mu(u) du \\ &= c_\mu \int_{[0,1]^r} \frac{1}{\theta^k} L_{a_{k-1}} \dots L_{a_0} T_0 e_\theta(xu_r) w_\mu(u) du \\ &= c_\mu \int_{[0,1]^r} \frac{1}{\theta^k} T_k L_{a_{k-1}} \dots L_{a_0} e_\theta(xu_r) w_\mu(u) du, \end{aligned}$$

which prove the result. ■

Example 4 : $r = 2, w = -1, \theta = i, \mu = (0, \alpha)$

Since $a_0 = 0$ then we can remove the index $i = 0$ in the representation of the function j_μ which gives

$$j_\mu(x) = j_\alpha(x) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \int_0^1 \cos(xu) (1-u^2)^{\alpha-\frac{1}{2}} du.$$

On the other hand

$$\begin{aligned} E_\mu(x) &= E_\alpha(x) = \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \int_0^1 \left[T_0 e^{ixu} + \frac{1}{i} T_1 \frac{d}{dx} e^{ixu} \right] (1-u^2)^{\alpha-\frac{1}{2}} du \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \int_0^1 [T_0 e^{ixu} + u T_1 e^{ixu}] (1-u^2)^{\alpha-\frac{1}{2}} du \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \int_0^1 \left[T_0 e^{ixu} + T_1 \frac{1}{x} (xu) e^{ixu} \right] (1-u^2)^{\alpha-\frac{1}{2}} du \end{aligned}$$

To find the classical form we can write

$$\begin{aligned} E_\alpha(x) &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \int_0^1 \left[\frac{e^{ixu} + e^{-ixu}}{2} + u \frac{e^{ixu} - e^{-ixu}}{2} \right] (1-u^2)^{\alpha-\frac{1}{2}} du \\ &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})} \int_{-1}^1 e^{ixu} (1+u)(1-u^2)^{\alpha-\frac{1}{2}} du. \end{aligned}$$

Example 5 : $r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3})$.

We have

$$j_\mu(x) = j_v(x) = 3 \frac{\Gamma(v + \frac{2}{3})}{\Gamma(v) \Gamma(\frac{2}{3})} \int_0^1 \cos_3(xu) (1-u^3)^{v-1} u du.$$

Then

$$E_v(x) = 3 \frac{\Gamma(v + \frac{2}{3})}{\Gamma(v) \Gamma(\frac{2}{3})} \int_0^1 \left[T_0 e^{\theta xu} + \frac{1}{\theta} T_1 \frac{d}{dx} e^{\theta xu} + \frac{1}{\theta^2} T_2 \left(\frac{d}{dx} + \frac{3v}{x} \right) \frac{d}{dx} e^{\theta xu} \right] (1-u^3)^{v-1} u du$$

which can be written in the following form

$$\begin{aligned} E_v(x) &= 3 \frac{\Gamma(v + \frac{2}{3})}{\Gamma(v) \Gamma(\frac{2}{3})} \times \\ &\int_0^1 \left[T_0 \frac{1}{x} (xu) e^{\theta xu} + T_1 \frac{1}{x^2} (xu)^2 e^{\theta xu} + T_2 \frac{1}{x^3} (xu)^3 e^{\theta xu} + \frac{3v}{\theta} T_2 \frac{1}{x^3} (xu)^2 e^{\theta xu} \right] (1-u^3)^{v-1} u du. \end{aligned}$$

4 Riemann–Liouville transform

Considering the r -Riemann Liouville operators of the form

$$R_\alpha g(x) = \int_0^1 g(xt) (1-t)^{\alpha-1} dt.$$

The integral representation in Theorem 1 of the Bessel function of vector index can be rewritten as follows

$$j_\mu(x) = c_\mu \int_{[0,1]^r} \cos_r(xu_r) w_\mu(u) du = c_\mu \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) \cos_r(x). \quad (8)$$

Now we study the inverse of R_α . We begin by state:

Theorem 3 For k integer and $0 < \alpha < 1$ we have

$$R_{k+\alpha}^{-1} g(x) = \frac{r^2}{\Gamma(k+1)\Gamma(\alpha)\Gamma(1-\alpha)} x^{r-1} \left(\frac{1}{rx^{r-1}} \frac{d}{dx} \right)^{k+1} \int_0^x g(u) (x^r - u^r)^{-\alpha} u^{(k+\alpha)r} du.$$

Proof. The operator R_α can be take the form

$$R_\alpha g(x) = \frac{1}{x^{1+r(\alpha-1)}} \int_0^x g(u) [x^r - u^r]^{\alpha-1} du$$

and for $0 < \alpha < 1$ admits as inverse :

$$R_\alpha^{-1}g(t) = \frac{r}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t g(x) [t^r - x^r]^{-\alpha} x^{\alpha r} dx$$

which is shown as follows

$$\begin{aligned} R_\alpha^{-1}R_\alpha g(t) &= \frac{r}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t R_\alpha g(x) [t^r - x^r]^{-\alpha} x^{\alpha r} dx \\ &= \frac{r}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \left[\frac{1}{x^{1+r(\alpha-1)}} \int_0^x g(u) [x^r - u^r]^{\alpha-1} du \right] [t^r - x^r]^{-\alpha} x^{\alpha r} dx \\ &= \frac{r}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \left[\int_u^t [x^r - u^r]^{\alpha-1} [t^r - x^r]^{-\alpha} x^{r-1} dx \right] g(u) du \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \left[\int_{u^r}^{t^r} [y - u^r]^{\alpha-1} [t^r - y]^{-\alpha} dy \right] g(u) du \\ &= \frac{d}{dt} \int_0^t g(u) du = g(t). \end{aligned}$$

Therefore

$$R_\alpha^{-1} = \frac{r}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} x^{1-\alpha r} R_{1-\alpha} x^{\alpha r}.$$

For an integer k we have the following relation

$$\frac{r}{k!} x^{r-1} \left(\frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} \int_0^x g(u) (x^r - u^r)^k du = g(x)$$

then

$$R_{k+1}^{-1} = \frac{r}{k!} x^{r-1} \left(\frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} x^{1+kr}.$$

On the other hand

$$\begin{aligned} R_{k+\alpha} g(x) &= \frac{1}{x^{1+(k+\alpha-1)r}} \int_0^x (x^r - u^r)^{k+\alpha-1} du \\ &= \frac{1}{x^{1+(k+\alpha-1)r}} \int_0^x (x^r - u^r)^k (x^r - u^r)^{\alpha-1} du \\ &= \frac{1}{x^{kr}} \frac{1}{x^{1+(\alpha-1)r}} \int_0^x \left[\frac{d}{du} \int_0^u (x^r - s^r)^k ds \right] (x^r - u^r)^{\alpha-1} du \end{aligned}$$

then

$$R_{k+\alpha} = \frac{1}{x^{kr}} R_\alpha \frac{d}{dx} x^{1+rk} R_{k+1}.$$

So we have

$$\begin{aligned} R_{k+\alpha}^{-1} &= R_{k+1}^{-1} \frac{1}{x^{1+rk}} \left(\frac{d}{dx} \right)^{-1} R_\alpha^{-1} x^{kr} \\ R_{k+\alpha}^{-1} &= \frac{r^2}{\Gamma(k+1)\Gamma(\alpha)\Gamma(1-\alpha)} x^{r-1} \left(\frac{1}{r x^{r-1}} \frac{d}{dx} \right)^{k+1} x^{1-\alpha r} R_{1-\alpha} x^{(k+\alpha)r}. \end{aligned}$$

The result is then established. ■

5 Hilbertian structure

We equipped the space F of complex valued functions by the hermitian scalar product given by:

$$\langle f, g \rangle_a = \int_0^\infty \left[\sum_{m=0}^{r-1} f(w^m t) \overline{g(w^m t)} \right] t^a dt \quad (9)$$

where a is a suitable positive real number.

We need to list some properties of the resulting hermitian structure. We begin by showing that the projectors T_i given by (2) are then symmetric . Indeed

$$\begin{aligned} \langle f, T_i g \rangle_a &= \int_0^\infty \left[\sum_{m=0}^{r-1} f(w^m t) \overline{T_i g(w^m t)} \right] t^a dt \\ &= \frac{1}{r} \int_0^\infty \left[\sum_{m=0}^{r-1} \sum_{k=0}^{r-1} \overline{w^{ik}} f(w^m t) \overline{g(w^{m+k} t)} \right] t^a dt \\ &= \frac{1}{r} \int_0^\infty \left[\sum_{k=0}^{r-1} \sum_{m'=0}^{r-1} w^{-ik} f(w^{m'-k} t) \overline{g(w^{m'} t)} \right] t^a dt \\ &= \frac{1}{r} \int_0^\infty \left[\sum_{m'=0}^{r-1} \sum_{k'=0}^{r-1} w^{ik'} f(w^{m'+k'} t) \overline{g(w^{m'} t)} \right] t^a dt \\ &= \langle T_i f, g \rangle_a . \end{aligned}$$

As a direct consequence we notice that if $i \neq j$ and $f \in F_i, g \in F_j$ then we have

$$\langle f, g \rangle = \langle T_i f, T_j g \rangle = \langle T_j T_i f, g \rangle = 0.$$

One can also verify the following identities

$$\left\langle f, \frac{1}{x} g \right\rangle_a = \left\langle \frac{1}{\bar{x}} f, g \right\rangle_a$$

and

$$\langle f, xg \rangle_a = \langle \bar{x}f, g \rangle_a .$$

Hence we have:

Proposition 4 *Let f and g be two complex valued functions such as*

$$\lim_{x \rightarrow 0, \infty} \left[f(x) \overline{g(x)} x^a \right] \quad \text{is finite.}$$

Then

$$\left\langle \frac{d}{dx} f, g \right\rangle_a = - \left\langle f, \left(\frac{d}{dx} + \frac{a}{\bar{x}} \right) g \right\rangle_a .$$

Proof. We have

$$\begin{aligned}
\left\langle \frac{d}{dx} f, g \right\rangle_a &= \sum_{m=0}^{r-1} \int_0^\infty \frac{df}{dx}(w^m x) \overline{g(w^m x)} x^a dx \\
&= \sum_{m=0}^{r-1} \int_0^\infty \frac{1}{w^m} \frac{d}{dx} f(w^m x) \overline{g(w^m x)} x^a dx \\
&= \left(\sum_{m=0}^{r-1} \frac{1}{w^m} \right) \left\{ \lim_{x \rightarrow \infty} [f(x) \overline{g(x)} x^a] - \lim_{x \rightarrow 0} [f(x) \overline{g(x)} x^a] \right\} \\
&\quad - \sum_{m=0}^{r-1} \int_0^\infty f(w^m x) \overline{\left[\frac{dg}{dx}(w^m x) + \frac{a}{w^m x} g(w^m x) \right]} x^a dx \\
&= - \left\langle f, \left(\frac{d}{dx} + \frac{a}{x} \right) g \right\rangle_a.
\end{aligned}$$

This is true because we have

$$\sum_{m=0}^{r-1} \frac{1}{w^m} = 0.$$

■

Now we are able to determine the adjoint of the r -extension of the Riemann Liouville operator and those related to the r - extension of Dunkl operator :

Proposition 5 For k integer and $0 < \alpha < 1$ the adjoint of the Riemann–Liouville operator is given by

$$R_\alpha^* g(u) = \int_1^\infty g(ut) [t^r - 1]^{\alpha-1} t^{a-1-r(\alpha-1)} dt$$

and

$$\begin{aligned}
R_{k+\alpha}^{*-1} g(\lambda) &= (-1)^{k+1} \frac{r^{1-k}}{\Gamma(k+1)\Gamma(\alpha)\Gamma(1-\alpha)} \lambda^{(k+1+\alpha)r-1} \\
&\times \int_1^\infty \left(\frac{d}{dx} \frac{1}{x^{r-1}} + \frac{a}{x^r} \right)^{k+1} g(\lambda x) (x^r - 1)^{\alpha-1} x^{a-2-r(\alpha-2)} dx.
\end{aligned}$$

Proof. In fact we have

$$\begin{aligned}
\langle R_\alpha f, g \rangle_a &= \sum_{m=0}^{r-1} \int_0^\infty R_\alpha f(w^m x) \overline{g(w^m x)} x^a dx \\
&= \sum_{m=0}^{r-1} \int_0^\infty \left[\int_0^x f(w^m u) [x^r - u^r]^{\alpha-1} du \right] \overline{g(w^m x)} x^{a-1-r(\alpha-1)} dx \\
&= \sum_{m=0}^{r-1} \int_0^\infty f(w^m u) \left[\int_u^\infty \overline{g(w^m x)} [x^r - u^r]^{\alpha-1} x^{a-1-r(\alpha-1)} dx \right] du \\
&= \sum_{m=0}^{r-1} \int_0^\infty f(w^m u) \overline{R_\alpha^* g(w^m u)} du \\
&= \langle f, R_\alpha^* g \rangle_a.
\end{aligned}$$

Therefore

$$\begin{aligned} R_\alpha^* g(u) &= u^{-a} \int_u^\infty g(x) [x^r - u^r]^{\alpha-1} x^{a-1-r(\alpha-1)} dx \\ &= \int_1^\infty g(ut) [t^r - 1]^{\alpha-1} t^{a-1-r(\alpha-1)} dt. \end{aligned}$$

Since we have

$$R_{k+\alpha}^{-1} = \frac{r^2}{\Gamma(k+1)\Gamma(\alpha)\Gamma(1-\alpha)} x^{r-1} \left(\frac{1}{rx^{r-1}} \frac{d}{dx} \right)^{k+1} x^{1-\alpha r} R_{1-\alpha} x^{(k+\alpha)r}$$

then

$$(R_{k+\alpha}^{-1})^* = R_{k+\alpha}^{*-1} = (-1)^{k+1} \frac{r^{1-k}}{\Gamma(k+1)\Gamma(\alpha)\Gamma(1-\alpha)} \bar{x}^{(k+\alpha)r} R_{1-\alpha}^* \bar{x}^{1-\alpha r} \left(\left(\frac{d}{dx} + \frac{a}{\bar{x}} \right) \frac{1}{\bar{x}^{r-1}} \right)^{k+1} \bar{x}^{r-1}$$

which leads to the result. ■

Proposition 6 *The corresponding adjoint of the r -extension Dunkl operator namely*

$$D_\mu = \frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k$$

is given by

$$D_\mu^* = - \left(\frac{d}{dx} + \frac{1}{\bar{x}} \sum_{k=0}^{r-1} (a - a_k) T_{k+1} \right),$$

where a is the real taking place in the definition of the inner product (9).

Proof. We performs the following calculation

$$\begin{aligned} & \left\langle \left(\frac{d}{dx} + \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k \right) f, g \right\rangle_a = \left\langle \frac{d}{dx} f, g \right\rangle_a + \left\langle \frac{1}{x} \sum_{k=0}^{r-1} a_k T_k f, g \right\rangle_a \\ &= - \left\langle f, \frac{d}{dx} g \right\rangle_a - \left\langle f, \frac{a}{\bar{x}} g \right\rangle_a + \left\langle f, \sum_{k=0}^{r-1} a_k T_k \frac{1}{\bar{x}} g \right\rangle_a \\ &= - \left\langle f, \frac{d}{dx} g \right\rangle_a - \left\langle f, \frac{1}{\bar{x}} \sum_{k=0}^{r-1} a T_{k+1} g \right\rangle_a + \left\langle f, \frac{1}{\bar{x}} \sum_{k=0}^{r-1} a_k T_{k+1} g \right\rangle_a \\ &= - \left\langle f, \left(\frac{d}{dx} + \frac{1}{\bar{x}} \sum_{k=0}^{r-1} (a - a_k) T_{k+1} \right) g \right\rangle_a. \end{aligned}$$

This proves the result. ■

Example 6 : $r = 2, w = -1, \theta = i, \mu = (0, \alpha)$

We choose $a = 2\alpha + 1$ then we get

$$\langle f, g \rangle = \int_0^\infty [f(t)g(t) + f(-t)g(-t)] t^{2\alpha+1} dt = \int_{-\infty}^\infty f(t)g(t) |t|^{2\alpha+1} dt.$$

On the other hand the Dunkl operator is given by

$$D_\alpha = \frac{d}{dx} + \frac{2\alpha + 1}{x} T_1$$

which implies

$$D_\alpha^* = - \left(\frac{d}{dx} + \frac{2\alpha + 1}{x} T_1 \right) = -D_\alpha.$$

Example 7 : $r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3})$

We choose $a = 3v$. The Dunkl operator is given by

$$D_v = \frac{d}{dx} + \frac{3v}{x} T_1$$

which implies

$$D_v^* = - \left(\frac{d}{dx} + \frac{3v}{x} T_1 \right) = -D_v.$$

We note that in general we have $D_\mu^* \neq -D_\mu$; the equality depends of a suitable choice of the real a .

6 Transmutation operator V_μ

An interesting topics is to seek an operator V_μ (see [5, 11] for the classical one case $r = 2$) which transforms $e^{\theta x}$ into $E_\mu(x)$. To make this section self containing we recall some properties shown early .

$$R_\alpha T_i = T_i R_\alpha, \quad T_i \frac{1}{x} = \frac{1}{x} T_{i-1}, \quad T_i x = x T_{i+1}, \quad T_{i+r} = T_i, \quad T_i^2 = T_i, \quad T_i T_j = 0 \text{ if } i \neq j$$

Theorem 4 *The transmutation kernel V_μ has the following form*

$$\begin{aligned} V_\mu = c_\mu T_0 \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) \\ + c_\mu \sum_{k=1}^{r-1} \sum_{j=0}^k \frac{P_j}{\theta^j} T_k \frac{1}{x^k} \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) x^{k-j}. \end{aligned} \quad (10)$$

where

$$P_{k-s} = \frac{1}{s!} \sum_{j=0}^s (-1)^{s-j} C_{s-j}^j \prod_{i=0}^{k-1} (a_i + i + j).$$

Proof. We start with the representation integral (8) and since we can write

$$L_{a_{k-1}} \dots L_{a_0} = \sum_{j=0}^k P_j \frac{1}{x^j} \left(\frac{d}{dx} \right)^{k-j}.$$

The constants P_j will be explained later. So we have

$$\begin{aligned} \left(\sum_{k=0}^{r-1} \frac{1}{\theta^k} T_k L_{a_{k-1}} \dots L_{a_0} \right) e_\theta(xu_r) &= \sum_{k=0}^{r-1} \frac{1}{\theta^k} T_k \left(\sum_{j=0}^k P_j \frac{1}{x^j} \left(\frac{d}{dx} \right)^{k-j} e_\theta(xu_r) \right) \\ &= \sum_{k=0}^{r-1} T_k \left(\sum_{j=0}^k \frac{P_j}{\theta^j} \frac{1}{x^j} u_r^{k-j} \right) e_\theta(xu_r). \end{aligned}$$

Hence

$$\begin{aligned} E_\mu(x) &= c_\mu \int_{[0,1]^r} \left(T_0 + \sum_{k=1}^{r-1} \frac{1}{\theta^k} T_k L_{a_{k-1}} \dots L_{a_0} \right) e_\theta(xu_r) w_\mu(u) du \\ &= c_\mu \int_{[0,1]^r} \left(T_0 + \sum_{k=1}^{r-1} T_k \left(\sum_{j=0}^k \frac{P_j}{\theta^j} \frac{1}{x^j} u_r^{k-j} \right) \right) e_\theta(xu_r) w_\mu(u) du. \end{aligned}$$

The transmutation operator V_μ is written as

$$\begin{aligned} V_\mu g(x) &= c_\mu \int_{[0,1]^r} \left(T_0 + \sum_{k=1}^{r-1} T_k \left(\sum_{j=0}^k \frac{P_j}{\theta^j} \frac{1}{x^j} u_r^{k-j} \right) \right) g(xu_r) w_\mu(u) du \\ &= c_\mu \left[T_0 \int_{[0,1]^r} g(xu_r) w_\mu(u) du + \sum_{k=1}^{r-1} \sum_{j=0}^k T_k \frac{P_j}{\theta^j} \frac{1}{x^j} \int_{[0,1]^r} g(xu_r) w_\mu(u) u_r^{k-j} du \right]. \end{aligned}$$

Note that

$$\int_{[0,1]^r} g(xu_r) w_\mu(u) du = \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) g(x),$$

then

$$\begin{aligned} \int_{[0,1]^r} g(xu_r) w_\mu(u) u_r^{k-j} du &= \frac{1}{x^{k-j}} \int_{[0,1]^r} g(xu_r) (xu_r)^{k-j} w_\mu(u) du \\ &= \frac{1}{x^{k-j}} \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) x^{k-j} g(x). \end{aligned}$$

Finally

$$\begin{aligned} V_\mu &= c_\mu T_0 \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) \\ &\quad + c_\mu \sum_{k=1}^{r-1} \sum_{j=0}^k \frac{P_j}{\theta^j} T_k \frac{1}{x^k} \prod_{i=0}^{r-1} \left(\frac{1}{x^{r-(i+1)}} R_{\alpha_i + \frac{i}{r} - 1} x^{r-(i+1)} \right) x^{k-j}. \end{aligned}$$

In this representation we can remove the components associated with indices i such that $a_i = 0$.

To explicate the constants P_j we recall that

$$L_a = x^{-a} \frac{d}{dx} x^a = \frac{d}{dx} + \frac{a}{x}.$$

We check that

$$\frac{1}{x^k} \prod_{i=0}^{k-1} \left(x \frac{d}{dx} + a_i + i \right) = L_{a_{k-1}} \dots L_{a_0}.$$

The use of the modified identity proven by Klushantsev [9]

$$\frac{1}{x^k} \prod_{j=0}^{k-1} \left(x \frac{d}{dx} + a_j + j \right) = \sum_{j=0}^k P_j \frac{1}{x^j} \left(\frac{d}{dx} \right)^{k-j},$$

where

$$P_{k-s} = \frac{1}{s!} \sum_{j=0}^s (-1)^{s-j} C_{s-j}^j \prod_{i=0}^{k-1} (a_i + i + j)$$

leads to the result. ■

Remark 2 : Thanks to relation (10) of the Theorem 4 , we can compute the inverse of the operator V_μ but being given the complicity of writing we just give V_μ^{-1} in the case of the following example.

Example 8 : $r = 2, w = -1, \theta = i, \mu = (0, \alpha)$

We have

$$R_{\alpha+\frac{1}{2}} g(x) = \int_0^1 g(xu) (1-u^2)^{\alpha-\frac{1}{2}} du,$$

then

$$c_\mu = c_\alpha = 2 \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2}) \Gamma(\frac{1}{2})}.$$

The operator $V_\mu = V_\alpha$ is then

$$V_\alpha = c_\alpha \left[T_0 R_{\alpha+\frac{1}{2}} + T_1 \frac{1}{x} R_{\alpha+\frac{1}{2}} x \right]$$

and we obtain

$$V_\alpha^{-1} = \frac{1}{c_\alpha} \left[R_{\alpha+\frac{1}{2}}^{-1} T_0 + \frac{1}{x} R_{\alpha+\frac{1}{2}}^{-1} x T_1 \right].$$

That can be justified as follows

$$\begin{aligned} V_\alpha^{-1} V_\alpha &= R_{\alpha+\frac{1}{2}}^{-1} T_0 R_{\alpha+\frac{1}{2}} + \frac{1}{x} R_{\alpha+\frac{1}{2}}^{-1} x T_1 \frac{1}{x} R_{\alpha+\frac{1}{2}} x \\ &= R_{\alpha+\frac{1}{2}}^{-1} R_{\alpha+\frac{1}{2}} T_0 + \frac{1}{x} R_{\alpha+\frac{1}{2}}^{-1} x \frac{1}{x} T_0 R_{\alpha+\frac{1}{2}} x \\ &= T_0 + \frac{1}{x} R_{\alpha+\frac{1}{2}}^{-1} R_{\alpha+\frac{1}{2}} T_0 x \\ &= T_0 + T_1 = id. \end{aligned}$$

On the other hand the adjoint of V_α take the form:

$$\begin{aligned} V_\alpha^* &= c_\alpha \left[R_{\alpha+\frac{1}{2}}^* T_0^* + x R_{\alpha+\frac{1}{2}}^* \frac{1}{x} T_1^* \right] \\ &= c_\alpha \left[R_{\alpha+\frac{1}{2}}^* T_0 + x R_{\alpha+\frac{1}{2}}^* \frac{1}{x} T_1 \right] \end{aligned}$$

where

$$R_{\alpha+\frac{1}{2}}^* g(u) = \int_1^\infty g(ut) [t^2 - 1]^{\alpha-\frac{1}{2}} t dt,$$

and

$$V_\alpha^{*-1} = \frac{1}{c_\alpha} \left[T_0 R_{\alpha+\frac{1}{2}}^{*-1} + T_1 x R_{\alpha+\frac{1}{2}}^{*-1} \frac{1}{x} \right].$$

Example 9 : $r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3})$

We have

$$R_v g(x) = \int_0^1 g(xu) (1 - u^3)^{v-1} du$$

$$c_\mu = c_v = 3 \frac{\Gamma(v + \frac{2}{3})}{\Gamma(v) \Gamma(\frac{2}{3})}.$$

The operator $V_\mu = V_v$ is given by

$$V_v = c_v \left[T_0 \frac{1}{x} R_v x + T_1 \frac{1}{x^2} R_v x^2 + T_2 \frac{1}{x^3} R_v x^3 + \frac{3v}{\theta} T_2 \frac{1}{x^3} R_v x^2 \right]$$

then its inverse is given by

$$V_v^{-1} = \frac{1}{c_v} \left[\frac{1}{x} R_v^{-1} x T_0 + \frac{1}{x^2} R_v^{-1} x^2 T_1 + \frac{1}{x^3} R_v^{-1} x^3 T_2 - \frac{3v}{\theta} \frac{1}{x^3} R_v^{-1} x^2 T_1 \right].$$

Since

$$\begin{aligned} V_v^{-1} V_v &= \frac{1}{x} R_v^{-1} x T_0 \frac{1}{x} R_v x + \frac{1}{x^2} R_v^{-1} x^2 T_1 \frac{1}{x^2} R_v x^2 + \frac{1}{x^3} R_v^{-1} x^3 T_2 \frac{1}{x^3} R_v x^3 \\ &\quad + \frac{3v}{\theta} \frac{1}{x^3} R_v^{-1} x^3 T_2 \frac{1}{x^3} R_v x^2 - \frac{3v}{\theta} \frac{1}{x^3} R_v^{-1} x^2 T_1 \frac{1}{x^2} R_v x^2 \\ &= T_0 + T_1 + T_2 + \frac{3v}{\theta} \frac{1}{x} T_1 - \frac{3v}{\theta} \frac{1}{x} T_1 = T_0 + T_1 + T_2 = id \end{aligned}$$

On the other hand

$$V_v^* = c_v \left[\bar{x} R_v^* \frac{1}{\bar{x}} T_0 + \bar{x}^2 R_v^* \frac{1}{\bar{x}^2} T_1 + \bar{x}^3 R_v^* \frac{1}{\bar{x}^3} T_2 + \frac{3v}{\theta} \bar{x}^2 R_v^* \frac{1}{\bar{x}^3} T_2 \right]$$

where

$$R_v^* g(u) = \int_1^\infty g(ut) [t^3 - 1]^{v-1} t^2 dt.$$

7 The operators D_μ and $\frac{d}{dx}$

In this section, we tackle the crucial subject concerning the research of functional spaces on which the following transmutation relation is valid

$$D_\mu V_\mu = V_\mu \frac{d}{dx}$$

The first idea that comes to mind is to verify that

$$D_\mu V_\mu x^n = V_\mu \frac{d}{dx} x^n, \quad \forall n \in \mathbb{N} \quad (11)$$

and when this last fact is true then the transmutation act on the space of entire function .

$$D_\mu V_\mu g(x) = V_\mu \frac{d}{dx} g(x), \quad g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

In fact we can permute each of the following operators

$$\frac{d}{dx}, R_\alpha, T_k, x^k, \frac{1}{x^k}$$

with the infinite sum $\sum_{n=0}^{\infty}$.

We will give two examples and we will constat that in the first, formula (11) is true but for the second it is false .

Example 10 : $r = 2, w = -1, \theta = i, \mu = (0, \alpha)$

We will prove that formula (11) is true in this case. For this we use the following result

$$R_\alpha x^n = \left(\int_0^1 (1-u^r)^{\alpha-1} u^n du \right) x^n = \frac{1}{r} \frac{\Gamma\left(\frac{n+1}{r}\right) \Gamma(\alpha)}{\Gamma\left(\alpha + \frac{n+1}{r}\right)} x^n = l_n^\alpha x^n,$$

then

$$\begin{aligned} D_\alpha V_\alpha x^{2n} &= c_\alpha \left(\frac{d}{dx} + \frac{2\alpha+1}{x} T_1 \right) \left(T_0 R_{\alpha+\frac{1}{2}} + T_1 \frac{1}{x} R_{\alpha+\frac{1}{2}} x \right) x^{2n} \\ &= c_\alpha \left(\frac{d}{dx} + \frac{2\alpha+1}{x} T_1 \right) l_{2n}^{\alpha+\frac{1}{2}} x^{2n} = c_\alpha l_{2n}^{\alpha+\frac{1}{2}} (2n) x^{2n-1} \end{aligned}$$

$$\begin{aligned} V_\alpha \frac{d}{dx} x^{2n} &= c_\alpha \left(T_0 R_{\alpha+\frac{1}{2}} + T_1 \frac{1}{x} R_{\alpha+\frac{1}{2}} x \right) (2n) x^{2n-1} \\ &= c_\alpha l_{2n}^{\alpha+\frac{1}{2}} (2n) x^{2n-1}. \end{aligned}$$

On the other hand

$$\begin{aligned} D_\alpha V_\alpha x^{2n+1} &= c_\alpha \left(\frac{d}{dx} + \frac{2\alpha+1}{x} T_1 \right) \left(T_0 R_{\alpha+\frac{1}{2}} + T_1 \frac{1}{x} R_{\alpha+\frac{1}{2}} x \right) x^{2n+1} \\ &= c_\alpha \left(\frac{d}{dx} + \frac{2\alpha+1}{x} T_1 \right) l_{2n+2}^{\alpha+\frac{1}{2}} x^{2n+1} = c_\alpha l_{2n+2}^{\alpha+\frac{1}{2}} [(2n+1) + (2\alpha+1)] x^{2n}, \end{aligned}$$

and

$$\begin{aligned} V_\alpha \frac{d}{dx} x^{2n+1} &= c_\alpha \left(T_0 R_{\alpha+\frac{1}{2}} + T_1 \frac{1}{x} R_{\alpha+\frac{1}{2}} x \right) (2n+1) x^{2n} \\ &= c_\alpha l_{2n}^{\alpha+\frac{1}{2}} (2n+1) x^{2n}. \end{aligned}$$

To show equality we use the identity

$$l_{2n}^{\alpha+\frac{1}{2}}(2n+1) = l_{2n+2}^{\alpha+\frac{1}{2}}[(2n+1) + (2\alpha+1)].$$

Example 11 : $r = 3, w = e^{i\frac{2\pi}{3}} = j, \theta = e^{i\frac{\pi}{3}}, \mu = (0, v - \frac{1}{3}, -\frac{2}{3})$

The transmutation operator is given by

$$V_v = c_v \left[T_0 \frac{1}{x} R_v x + T_1 \frac{1}{x^2} R_v x^2 + T_2 \frac{1}{x^3} R_v x^3 + \frac{3v}{\theta} T_2 \frac{1}{x^3} R_v x^2 \right].$$

The 3-extension of Dunkl operator takes the following form

$$D_v = \frac{d}{dx} + \frac{3v}{x} T_1$$

We check easily that

$$\left(\frac{d}{dx} + \frac{3v}{x} T_1 \right) V_v x^{3n} \neq V_v \frac{d}{dx} x^{3n}.$$

So the spaces of entire function seems not suitable for transmutation for all r except for the case $r = 2$.

In the following statement we show that the transmutation is true over the following suitable functional space.

Theorem 5 *Let g be a continuously differentiable function on an interval $[\frac{T}{2}, \frac{T}{2}]$ such that*

$$\sum_{n=-\infty}^{\infty} |c_n(g)| e^{\pi |n \operatorname{Im}(w^k)|} < \infty, \quad \forall k = 0 \dots r-1 \quad (12)$$

then we have

$$D_\mu V_\mu g(x) = V_\mu \frac{d}{dx} g(x).$$

Proof. Since we have

$$D_\mu V_\mu e^{\theta \mu x} = V_\mu \frac{d}{dx} e^{\theta \mu x}, \quad \forall \mu \in \mathbb{C} \Rightarrow D_\mu V_\mu e^{i \lambda x} = V_\mu \frac{d}{dx} e^{i \lambda x}, \quad \forall \lambda \in \mathbb{C}.$$

Let g be a continuously differentiable function on an interval $[\frac{T}{2}, \frac{T}{2}]$ then we have

$$g(x) = \sum_{n=-\infty}^{\infty} c_n(g) e^{\frac{2i\pi}{T} n x}, \quad \forall x \in \left[-\frac{T}{2}, \frac{T}{2} \right].$$

The coefficients $c_n(g)$ so called the Fourier coefficients of g , defined by the formula

$$c_n(g) = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{\frac{2i\pi}{T} n t} dt.$$

We have

$$\sum_{n=-\infty}^{\infty} |c_n(g)| < \infty.$$

The action of the operator T_k at the function g shows in the Fourier series a terms of the form

$$e^{\frac{2i\pi}{T}nw^k t}, \quad w = e^{\frac{2i\pi}{r}}, \quad 0 \leq k \leq r-1.$$

As

$$\left| e^{\frac{2i\pi}{T}nw^k t} \right| \leq e^{\pi |n \operatorname{Im}(w^k)|}.$$

So if we impose the condition of normal convergence

$$\sum_{n=-\infty}^{\infty} |c_n(g)| e^{\pi |n \operatorname{Im}(w^k)|} < \infty, \quad \forall k = 0 \dots r-1$$

we say that

$$D_\mu V_\mu g(x) = V_\mu \frac{d}{dx} g(x).$$

■

Now we understand why this relationship is verified for x^n in the cas $r = 2$ because $w = -1$ and then $\operatorname{Im}(w^k) = 0$.

8 r-extension of Dunkl transform

Before anything let us introduce the integral transform of Laplace type given for $\theta = i\frac{\pi}{r}$ by :

$$\mathcal{L}_\theta g(\lambda) = \int_0^\infty e^{\theta t \lambda} g(t) dt.$$

Proposition 7 *The inversion formula of \mathcal{L}_θ is given by*

$$\mathcal{L}_\theta^{-1} g(x) = \frac{1}{2\pi i \bar{\theta}} \lim_{T \rightarrow \infty} \int_{-c\bar{\theta} - i\bar{\theta}T}^{-c\bar{\theta} + i\bar{\theta}T} e^{-\theta x s} g(s) ds$$

which is valid for any function of exponential type $\alpha < c$.

Proof. To prove this formula, given a function g of exponential type $\alpha < c$. Then there exists $M > 0$ such that

$$|g(t)| \leq M e^{\alpha t}, \quad \forall t \in \mathbb{R}.$$

If $s = -c\bar{\theta} + iy\bar{\theta}$ where $c > \alpha$ and $y > 0$ we get

$$\mathcal{L}_\theta g(s) = \int_0^\infty e^{\theta t(-c\bar{\theta} + is\bar{\theta})} g(t) dt = \int_0^\infty e^{iyt} e^{-ct} g(t) dt.$$

Therefore

$$|\mathcal{L}_\theta g(s)| \leq \int_0^\infty e^{-ct} |g(t)| dt \leq \int_0^\infty e^{(\alpha-c)t} dt < \infty.$$

So we have

$$\begin{aligned} \mathcal{L}_\theta^{-1} \mathcal{L}_\theta g(x) &= \frac{1}{2\pi i \bar{\theta}} \lim_{T \rightarrow \infty} \int_{-c\bar{\theta} - i\bar{\theta}T}^{-c\bar{\theta} + i\bar{\theta}T} e^{-\theta x s} \mathcal{L}_\theta g(s) ds \\ &= e^{cx} \left[\frac{1}{2\pi} \lim_{T \rightarrow \infty} \int_{-T}^T e^{-iyx} \left(\int_0^\infty e^{iyt} e^{-ct} g(t) dt \right) dy \right] \\ &= e^{cx} e^{-cx} g(x) = g(x), \quad \forall x \in \mathbb{R}. \end{aligned}$$

which prove the result ■

Definition 3 For $a > 0$, we define the r -extension of the Dunkl transform associated with the vector $\mu = (\alpha_0, \alpha_1, \dots, \alpha_{r-1})$ as follows

$$\mathcal{F}_\mu g(\lambda) = \langle g, E_\mu(\lambda x) \rangle_a = \int_0^\infty \left[\sum_{m=0}^{r-1} g(w^m t) \overline{E_\mu(w^m \lambda t)} \right] t^a dt$$

where E_μ denote the r -Dunkl kernel (5).

Taking account of the fact that $E_\mu(\lambda x) = V_\mu e^{\theta \lambda x}$ then we can write

$$\mathcal{F}_\mu g(\lambda) = \langle g, E_\mu(\lambda x) \rangle_a = \left\langle g, V_\mu e^{\theta \lambda x} \right\rangle_a = \left\langle V_\mu^* g, e^{\theta \lambda x} \right\rangle_a.$$

The integral transform associated with $\mu = (0, -\frac{1}{r}, \dots, -\frac{r-1}{r})$ is given by

$$\mathcal{F}_r g(\lambda) = \left\langle g, e^{\theta \lambda x} \right\rangle_0.$$

This operator coincide with the Laplace transform and Fourier transform respectively for $r = 1$ and $r = 2$.

We deduce that

$$\mathcal{F}_\mu g(\lambda) = \left\langle V_\mu^* g, e^{\theta \lambda x} \right\rangle_a = \left\langle |x|^a V_\mu^* g, e^{\theta \lambda x} \right\rangle_0.$$

Therefore

$$\mathcal{F}_\mu = \mathcal{F}_r |x|^a V_\mu^*.$$

Proposition 8 Let g be a function of exponential type belongs in F_{r-k} the subspace defined by (1) then we have

$$\mathcal{F}_\mu^{-1} g(\lambda) = \frac{1}{r} V_\mu^{*-1} |x|^{-a} \mathcal{L}_\theta^{-1} g(\lambda).$$

Proof. We write the transformation \mathcal{F}_r as follows

$$\mathcal{F}_r g(\lambda) = \left\langle g, e^{\theta \lambda x} \right\rangle_0 = \int_0^\infty \left(\sum_{m=0}^{r-1} g(w^m t) e^{w^m \theta t \lambda} \right) dt$$

then

$$\begin{aligned} \mathcal{F}_r T_k g(\lambda) &= \int_0^\infty \left(\sum_{m=0}^{r-1} T_k g(w^m t) e^{w^m \theta t \lambda} \right) dt \\ &= \int_0^\infty \left(\sum_{m=0}^{r-1} w^{(r-k)m} e^{w^m \theta t \lambda} \right) T_k g(t) dt \\ &= r T_{r-k} \mathcal{L}_\theta T_k g(\lambda). \end{aligned}$$

Furthermore we have

$$\mathcal{F}_\mu T_k = \mathcal{F}_r |x|^a V_\mu^* T_k = \mathcal{F}_r T_k |x|^a V_\mu^* = r T_{r-k} \mathcal{L}_\theta T_k |x|^a V_\mu^* = r T_{r-k} \mathcal{L}_\theta |x|^a V_\mu^* T_k,$$

then

$$\mathcal{F}_\mu = \sum_{k=0}^{r-1} \mathcal{F}_\mu T_k = r \sum_{k=0}^{r-1} T_{r-k} \mathcal{L}_\theta |x|^a V_\mu^* T_k,$$

which implies

$$T_m \mathcal{F}_\mu = r T_m \sum_{k=0}^{r-1} T_{r-k} \mathcal{L}_\theta |x|^a V_\mu^* T_k = r T_m \mathcal{L}_\theta |x|^a V_\mu^* T_{r-m}.$$

Notice that

$$\mathcal{F}_\mu : F_k \rightarrow F_{r-k}.$$

In the space F_k we have the following equality

$$\mathcal{F}_\mu = r T_{r-k} \mathcal{L}_\theta |x|^a V_\mu^*.$$

This leads to the result. ■

Proposition 9 *If $D_\mu^* = -D_\mu$ then we have*

$$\mathcal{F}_\mu D_\mu g(\lambda) = -\theta \lambda \mathcal{F}_\mu g(\lambda).$$

Proof. In fact

$$\begin{aligned} \mathcal{F}_\mu D_\mu^* g(\lambda) &= \langle D_\mu^* g, E_\mu(\lambda x) \rangle_a = \langle D_\mu^* g, V_\mu e^{\theta \lambda x} \rangle_a \\ &= \langle g, D_\mu V_\mu e^{\theta \lambda x} \rangle_a = \left\langle g, V_\mu \frac{d}{dx} e^{\theta \lambda x} \right\rangle_a \\ &= \theta \lambda \langle g, E_\mu(\lambda x) \rangle_a = \theta \lambda \mathcal{F}_\mu g(\lambda). \end{aligned}$$

In the case $D_\mu^* = -D_\mu$ we obtain the result.

Note that the function $x \mapsto e^{\theta \lambda x}$ is a continuously differentiable function which satisfies (12). ■

Epilogue

We just built an r -extension of Dunkl operator focusing on examples. This approach is very positive and encourages researchers to determine adequate harmonic analysis and especially look for applications.

In forthcoming papers we will study in great detail the associated heat and wave equations.

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